ECOM-G314: Econometrics 1
Second Retake

12 January 2018

- This exam has three parts and four pages. Start each part on a new sheet of paper (one sheet comprises four A4-sized pages). Don't copy the instructions!
- There are no negative points.
- Good luck!

1 Empirical (16 points, start on a new sheet of paper)

We consider the regression of (the logarithm of) labour (number of workers) on an intercept, (the logarithm of) wages (in 1000 EUR), (the logarithm of) capital (total fixed assets in 1,000,000 EUR), and (the logarithm of) output (value added in 1,000,000 EUR). We consider the linear model, i.e.

$$(LIN) \qquad labour_i = \beta_1 + wage_i \cdot \beta_2 + output_i \cdot \beta_2 + capital_i \cdot \beta_3 + \varepsilon_i$$

and the log-linear model, i.e.

$$(LOG) \qquad log(labour_i) = \beta_1 + log(wage_i) \cdot \beta_2 + log(output_i) \cdot \beta_2 + log(capital_i) \cdot \beta_3 + \eta_i.$$

The outputs of the regressions are

LIN	**LIN**	**LIN**
term	estimate	standard error
(Intercept)	280	20
wage	-7	0.5
output	15	0.3
capital	-4	0.27

and

LOG	**LOG**	**LOG**	**LOG**
term	estimate	standard error	p-value
(Intercept)	6	0.25	$3 \cdot 10^{-100}$
log(wage)	-1	0.05	$1 \cdot 10^{-10}$
log(output)	1	0.025	0.001
log(capital)	-0.04	0.2	0.075

Interpretation

- 1) **Interpret** the coefficient pertaining to wage in the linear model. (2 points for choosing the correct answer)
  - (A) For a ceteris paribus change in wage demand of workers by 1000 EUR, the labour demand of firms decreases by 7 workers.
  - (B) For a ceteris paribus change in wage demand of workers by 1%, the labour demand of firms decreases by 7%.
  - (C) For a ceteris paribus change in wage demand of workers by 1 EUR, the labour demand of firms decreases by 7 workers.
  - (D) For a ceteris paribus change in wage demand of workers by 1000 EUR, the labour demand of firms decreases by 7000 workers.
- 2) **Interpret** the coefficient pertaining to output in the log-linear model. (2 points for choosing the correct answer)

- (A) For a ceteris paribus change in output by 1%, the labour demand increases by 1%.
- (B) For a ceteris paribus change in log(output) by 1%, the log(labour) increases by 1%.
- (C) For a ceteris paribus change in output by 1,000,000 EUR, the logarithm of the labour demand increases by log(1,000,000) (≈ 14) workers.
- (D) For a ceteris paribus change in log(output) by log(1,000,000) EUR year, the logarithm of the labour demand increases by log(1,000,000) (≈ 14) workers.

- 3) The R^2 of the linear model is 93% while the R^2 of the log-linear model is only 84%. Therefore, the linear model should be preferred. **True or false?** (1 point)
  - If 3) correct: **Define the $\mathbf{R^2}$** in at most one sentence or formula. (1 point)
- 4) What is the **value of the t-statistic** pertaining to the intercept in the linear model? (1 point)

Testing

Analyzing the regression residuals is arguably the most important step in regression analysis.

- 5) **Define** (in at most one sentence or formula) the Type 1 and Type 2 error of a test. (2 points)
- 6) The H_0 of the **Breusch-Pagan test** is that the errors are homoskedastic and the alternative hypothesis is that the errors are heteroskedastic. **True or false**? (1 point)
  - If 6) correct: 6.1) When conducting inference with heteroskedastic errors, it is invalid to use unadjusted standard-errors for testing. This is due to the fact that the Gauß-Markov assumptions are not satisfied in this case. **True or false**? (1 point)
  - If 6) correct: 6.2) The Breusch-Pagan test in the (LIN) model is based on the (1 point for choosing the correct answer)
    * (A) R^2 of the regression of squared residuals on wage, output, and capital.
    * (B) R^2 of the regression of squared residuals on an intercept, wage, output, and capital.
    * (C) R^2 of the regression of residuals on wage, output, and capital.
    * (D) R^2 of the regression of residuals on an intercept, wage, output, and capital .
- 7) You **test** at significance level 5% the H_0 that the **coefficient pertaining to log_capital** is zero. (2 points)
  - (A) Reject the H_0: The coefficient is significantly different from zero.
  - (B) Reject the H_0: The coefficient is not significantly different from zero.
  - (C) Do not reject the H_0: The coefficient is significantly different from zero.
  - (D) Do not reject the H_0: The coefficient is not significantly different from zero.
- 8) PE-test: How can you **test the H_0 of no linear influence** in the logarithmic model? (2 point)

2 Gauß-Markov (19 points, start on a new sheet of paper)

- 1) State the four Gauß-Markov assumptions (in words and formula). (4 points)
- 2) The OLS estimator b is a linear function of the regressor matrix X. **True or false?** (1 point)
- 3) The population parameter β is unobserverd, while the OLS estimator $b = (X'X)^{-1}X'y$ is, as function of the data, observed. **True or false?** (1 point)
- 4) What is the **dimension** of $(X'X)^{-1}X'$? (1 point)
- 5) If one takes the expectation of the OLS estimator, one obtains a random variable. **True or false?** (1 point)
  - If 5) correct: Give the definition of an unbiased estimator. (1 point)

- 6) Which of the assumptions you stated in (1) are not required to prove unbiasedness of the OLS estimator? (2 points)
- 7) The GLS estimator $\hat{\beta} = \left(X'\Psi^{-1}X\right)^{-1} X'\Psi^{-1}y$ is unbiased under the assumptions you stated in (1). **True or false?** (1 point)

Assume that the true model $y = X\beta + \varepsilon$ is such that $\mathbb{E}\left(\varepsilon\varepsilon'\right) = \sigma^2\Psi$, where Ψ is an $(N \times N)$-dimensional positive definite and symmetric matrix (not equal to the identity matrix) and $\sigma^2 > 0$. Other than that, the assumptions are the same as you stated in (1).

- 8) Calculate the variance of the OLS estimator $b = \left(X'X\right)^{-1} X'y$ under the GLS assumptions. (3 points)
- 9) The OLS estimator is unbiased under the GLS assumptions. **True or false?** (1 point)
- 10) Give the definition of an efficient estimator. (2 points)
  - If 10) correct: The OLS estimator is efficient under the GLS assumptions. (1 point)

3 Omitted Variables (Start on a new sheet of paper) (16 points)

Lagrange Multiplier Test

Consider the linear model $y_i = x_i\beta + z_i\gamma + \varepsilon_i,\ i \in \{1, \ldots, N\}$, and assume that (A1) to (A5) hold. Both x_i and z_i are one-dimensional. We will derive the Lagrangian Multplier (LM) test for testing for omitted variables, i.e. we test the $H_0:\ \gamma = 0$. We know that the log-likelihood function is equal to

$$\begin{aligned} \log\left(L\left(\beta, \gamma, \sigma^2\right)\right) &= \sum_{i=1}^{N} \log\left(L_i\left(\beta, \gamma, \sigma^2\right)\right) \\ &= -\frac{N}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\varepsilon_i}{\sigma}\right)^2 \end{aligned}$$

- Formulate the **constrained maximization problem** of the log-likelihood function subject to the constraint $\gamma = 0$ in Lagrangian form.
  - 1) Write the log-likelihood function $\log\left(L\left(\beta, \gamma, \sigma^2\right)\right)$ **in terms of the observable quantities** y_i, x_i, and z_i. (1.5 point)
  - 2) Write down the Lagrangian form of the **constrained maximization problem** of the log-likelihood function subject to the constraint $\gamma = 0$. Use $H\left(\beta, \gamma, \sigma^2, \lambda\right)$ to denote the Lagrangian and λ to denote the Lagrangian Multiplier. (1 point)
  - 3) What dimension does λ have? (1 points)
- **Calculate the first order conditions (FOC)** of the Lagrangian with respect to (β, γ, λ)[1], i.e. set the partial derivatives equal to zero, denote the maximizing parameter values by $\left(\tilde{\beta}, \tilde{\gamma}, \tilde{\lambda}\right)$, and denote $\tilde{\varepsilon}_i = y_i - x_i'\tilde{\beta}$.
  - 4) Take the derviative of $H\left(\beta, \gamma, \sigma^2, \lambda\right) = -\frac{N}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2}\sum_{i=1}^{N}\left(\frac{y_i - x_i\beta - z_i\gamma}{\sigma}\right)^2 - \lambda\gamma$ with respect to β and set it equal to zero. (1 point)
  - 5) Take the derviative of $H\left(\beta, \gamma, \sigma^2, \lambda\right) = -\frac{N}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2}\sum_{i=1}^{N}\left(\frac{y_i - x_i\beta - z_i\gamma}{\sigma}\right)^2 - \lambda\gamma$ with respect to γ and set it equal to zero. (1 point)
  - 6) Take the derviative of the Lagrangian with respect to λ and set it equal to zero.(1 point)
  - 7) Explain why $\tilde{\varepsilon}_i = y_i - x_i'\tilde{\beta} = y_i - x_i'\tilde{\beta} - z_i\tilde{\gamma}$ holds. (1 point)

[1] It can be shown that the the partial derivative of the Lagrangian with respect to σ^2 is irrelevant to the result (and thus not part of this exam).

- We know that $\sqrt{N}\left(\frac{1}{N}\tilde{\lambda}_N\right) \xrightarrow[N\to\infty]{d} \mathcal{N}\left(0, \left(I^{22}(\tilde{\theta})\right)^{-1}\right)$ holds, where $I^{22}\left(\tilde{\theta}\right)^{-1}$ is obtained from

$$I(\tilde{\theta})^{-1} = \begin{pmatrix} I^{11}(\tilde{\theta}) & I^{12}(\tilde{\theta}) \\ I^{21}(\tilde{\theta}) & \boldsymbol{I^{22}(\tilde{\theta})} \end{pmatrix} = \begin{pmatrix} * & * \\ * & \left(\boldsymbol{I_{22}(\tilde{\theta}) - I_{21}(\tilde{\theta}) I_{11}(\tilde{\theta})^{-1} I_{12}(\tilde{\theta})}\right)^{\boldsymbol{-1}} \end{pmatrix}$$

and $I\left(\tilde{\theta}\right) = \frac{1}{N}\sum_{i=1}^{N} s_i\left(\tilde{\theta}\right) s_i\left(\tilde{\theta}\right)'$ and $s_{1,i}\left(\tilde{\theta}\right) = \frac{\partial \log\left(L_i\left(\tilde{\beta},\tilde{\gamma},\tilde{\sigma}^2\right)\right)}{\partial \beta}$, $s_{2,i}\left(\tilde{\theta}\right) = \frac{\partial \log\left(L_i\left(\tilde{\beta},\tilde{\gamma},\tilde{\sigma}^2\right)\right)}{\partial \gamma}$, and $\tilde{\theta} = \left(\tilde{\beta}, \tilde{\gamma}, \tilde{\sigma}^2, \tilde{\lambda}\right)$.

  - 8) It follows from the above that (2 points)
    - (A) $\xi_{LM} = \frac{1}{N}\tilde{\lambda}_N' I^{22}\left(\tilde{\theta}\right)\tilde{\lambda}_N$ converges towards a χ^2_1 distribution.
    - (B) $\xi_{LM} = \sqrt{\frac{1}{N}}\tilde{\lambda}_N' I^{22}\left(\tilde{\theta}\right)\tilde{\lambda}_N$ converges towards a χ^2_1 distribution.
    - (C) $\xi_{LM} = \frac{1}{N}\tilde{\lambda}_N' \left(I^{22}\left(\tilde{\theta}\right)\right)^{-1}\tilde{\lambda}_N$ converges towards a χ^2_1 distribution.
    - (D) $\xi_{LM} = \sqrt{\frac{1}{N}}\tilde{\lambda}_N' \left(I^{22}\left(\tilde{\theta}\right)\right)^{-1}\tilde{\lambda}_N$ converges towards a χ^2_1 distribution.

- It can be shown that $\xi_{LM} = \iota' S\left[S'S\right]^{-1} S'\iota$, where $S = \begin{pmatrix} s_1\left(\tilde{\theta}\right)' \\ \vdots \\ s_i\left(\tilde{\theta}\right)' \\ \vdots \\ s_N\left(\tilde{\theta}\right)' \end{pmatrix}$ and ι is a vector of ones of appropriate dimension.

  - 9) This expression is equal to N times the uncentered R^2 of the auxiliary regression of ι on the columns of S. True or false? (1 point)
  - 9.1) If 9) correct: Derive that $\iota' S\left[S'S\right]^{-1} S'\iota$ is equal to the (un-)centered R^2. (2 points)

General questions regarding omitted variables

Consider the same model as in the last question, i.e. $y_i = x_i\beta + z_i\gamma + \varepsilon_i, \quad i \in \{1, \ldots, N\}$ and (A1) to (A5) hold. Furthermore, $\gamma \neq 0$, and x_i and z_i are correlated.

- 10) Which of the following estimators are unbiased for β? $b^{(S)} = (X'X)^{-1} X'y$ and/or $b^{(S)} = (X'X)^{-1} X'y$? Or none of them? (1.5 points)
- 10.1) If 10) correct: Explain your answer. (1 point)
- 10.2) If 10) correct: Under which condition(s) are they unbiased? (1 point)

Your points will be multiplied with factor 1.19.