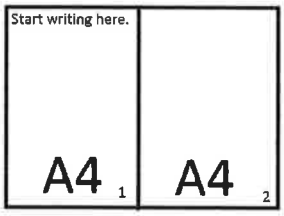
Advanced Econometrics I: Principles of Econometrics
First Retake Exam

30 January 2017

- This exam has three pages and parts.
- Don't copy the instructions.
- Start each part on a new sheet of paper (one sheet comprises four A4-sized pages), and use them in A3 format (which makes correcting your exam easier).
- Good luck!



1 Gauß-Markov Theorem (12 points)

Prove the following version of the Gauß-Markov Theorem: The estimator $Lb = L\left(X'X\right)^{-1}X'y$, where $L \in \mathbb{R}^{s\times K}$ is the best linear unbiased estimator for $L\beta$ in the linear model $y = X\beta + \varepsilon$, where $X \in \mathbb{R}^{N\times K}$, $N \geq K$, is non-stochastic and of full column rank, $\mathbb{E}\left(\varepsilon\right) = 0$, and the error terms are homoskedastic and uncorrelated.

1.1 Preliminary questions Statement and definitions (4,5 points)

1. (1 point) Let $\hat{\theta}$ be an estimator, i.e. a random variable from the sample space to the parameter space, for the population parameter θ. Give the formal definition of an unbiased estimator.
2. (1 point) We consider the class of all unbiased estimators $\tilde{\theta}$ for the population parameter θ. Give the formal definition of the efficient estimator in this class of estimators.
3. (0,5 points) Under the assumptions above, $\mathbb{E}\left(\varepsilon\right) = 0$ implies $\mathbb{E}\left(\varepsilon|X\right) = 0$. True or false?
4. (0,5 points) The statement "$\mathbb{E}\left(\varepsilon\right) = 0$ implies $\mathbb{E}\left(\varepsilon|X\right) = 0$" is true in general (for stochastic X). True or false?
5. (0,5 points) In the case of a stochastic matrix X, the condition $\mathbb{E}\left(\varepsilon_i x_i'\right) = 0$ implies that $\mathbb{E}\left(\varepsilon|X\right) = 0$. True or false?
6. (1 point) Which of the following are valid for stating that the error terms are homoskedastic and uncorrelated? (The question only counts if ALL correct possibilities are indicated.)

   (a) $\mathbb{V}\left(\varepsilon\right) = \sigma^2 I_N$
   (b) $\mathbb{V}\left(\varepsilon\right) = \sigma^2 I_K$
   (c) $\mathbb{V}\left(\varepsilon_i\right) = \sigma^2$ for all i and $\mathbb{C}\left(\varepsilon_i, \varepsilon_j\right) = 0$ for $i \neq j$
   (d) $\mathbb{V}\left(\varepsilon_i\right) = \sigma_i^2$ for all i and $\mathbb{C}\left(\varepsilon_i, \varepsilon_j\right) = 0$ for $i \neq j$

1.2 Part 1 of the proof (3,5 points)

1. (1 point) Calculate the expectation of Lb.
2. (2 points) Consider the linear estimator $L\check{b} = Dy$. Derive the conditions on the matrix D such that a linear estimator $L\check{b} = Dy$ is unbiased for $L\beta$.
3. (0,5 points) What dimensions does D have?

1.3 Part 2 of the proof (4 points)

1. (2 points) Calculate the variance of Lb.
2. (2 points) Show that for matrices $D,\ X \in \mathbb{R}^{N\times K}$, $L \in \mathbb{R}^{s\times K}$satisfying $DX = L$, the decompositition

$$DD' = \left(L\left(X'X\right)^{-1}X'\right)\left(L\left(X'X\right)^{-1}X'\right)' + \left(D - \left[L\left(X'X\right)^{-1}X'\right]\right)\left(D - \left[L\left(X'X\right)^{-1}X'\right]\right)'$$

   holds.

2 F-test (12 points)

Consider the regression model

$$y = X_1\beta^{(1)} + X_2\beta^{(2)} + \varepsilon$$

satisfying assumptions (A1)-(A5). Furthermore, assume that $X_1 \in \mathbb{R}^{N\times(K-J)}$ and $X_2 \in \mathbb{R}^{N\times J}$ are non-stochastic and of full column rank. To fix notation, denote $b = \begin{pmatrix} b^{(1)} \\ b^{(2)} \end{pmatrix}$ the OLS estimator for $\beta = \begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \end{pmatrix}$ and e the corresponding OLS residual such that

$$y = X_1 b^{(1)} + X_2 b^{(2)} + e.$$

1. (1 point) How is the projection on the orthogonal complement of the space spanned by the columns of X_1 defined?
2. (2 points) Consider the projection matrix M_{X_1} defined above, define $e_{restr} := M_{X_1}y$, $\tilde{X}_2 := M_{X_1}X_2$ and apply M_{X_1} to the equation $y = X_1b^{(1)} + X_2b^{(2)} + e$. Calculate $M_{X_1}X_1b^{(1)}$ (0.5 points) and $M_{X_1}e$ (0.5 points) and explain your reasoning (0.5 points respectively).
3. (1 point) Consider $e_{restr}'e_{restr} := \left(\tilde{X}_2b^{(2)} + e\right)'\left(\tilde{X}_2b^{(2)} + e\right)$. Show that $e_{restr}'e_{restr} = \left(b^{(2)}\right)'\tilde{X}_2'\tilde{X}_2b^{(2)} + e'e$ and explain your calculations.
4. (3 points) From the Frisch-Waugh Theorem we know that $b^{(2)} = \left(\left(\tilde{X}_2\right)'\tilde{X}_2\right)^{-1}\left(\tilde{X}_2\right)'y$. Show that under the $H_0 : \beta^{(2)} = 0$, the equation $\tilde{X}_2b^{(2)} = \tilde{X}_2\left(\left(\tilde{X}_2\right)'\tilde{X}_2\right)^{-1}\left(\tilde{X}_2\right)'\varepsilon$ holds and explain your calculations (1.5 points).

   Furthermore (1.5 points), show that $\left(b^{(2)}\right)'\tilde{X}_2'\tilde{X}_2b^{(2)} = \varepsilon'P_{\tilde{X}_2}\varepsilon$, where $P_{\tilde{X}_2} := \tilde{X}_2\left(\left(\tilde{X}_2\right)'\tilde{X}_2\right)^{-1}\left(\tilde{X}_2\right)'$.
5. (3 points) Show that $\frac{1}{\sigma^2}e'e$ is equal to $\frac{1}{\sigma^2}\varepsilon'M_X\varepsilon$. Then, show that $\frac{1}{\sigma^2}\varepsilon'M_X\varepsilon$ and $\frac{1}{\sigma^2}\left[\varepsilon'P_{\tilde{X}_2}\varepsilon\right]$ are independent.
   (a) Hint: In order to show that $\frac{1}{\sigma^2}\varepsilon'M_X\varepsilon$ and $\frac{1}{\sigma^2}\left[\varepsilon'P_{\tilde{X}_2}\varepsilon\right]$ are independent, start by showing that $M_X\varepsilon$ and $P_X\varepsilon$ are uncorrelated, then use properties of normal distributions to show that $\frac{1}{\sigma^2}\varepsilon'M_X\varepsilon$ and $\frac{1}{\sigma^2}\left[\varepsilon'P_{\tilde{X}_2}\varepsilon\right]$ are independent.
6. (2 points) We know from 5) that $\frac{1}{\sigma^2}\varepsilon'M_X\varepsilon$ and $\frac{1}{\sigma^2}\left[\varepsilon'P_{\tilde{X}_2}\varepsilon\right]$ are independent.
   (a) (1 point) How are these quantities related to the terms in $f = \frac{\frac{\left(e_{restr}'e_{restr} - e'e\right)}{J}}{\frac{e'e}{N-K}}$?

(b) (1 point) What are the distributions of $\frac{1}{\sigma^2}\varepsilon' M_X \varepsilon$ and $\frac{1}{\sigma^2}\left[\varepsilon' P_{\tilde{X}_2}\varepsilon\right]$ and why does it follow from their independence that $f = \frac{\left(e'_{restr}e_{restr} - e'e\right)/J}{\frac{e'e}{N-K}}$ is $F_{J,N-K}$ distributed?

3 Short Questions (12 points + 1 bonus)

1. (3 points) We consider the linear regression model

$$y = X\beta + \varepsilon$$

where $X \in \mathbb{R}^{N\times K}$ is non-stochastic and of full rank., $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{V}(\varepsilon) = \sigma^2\Psi$, where $\sigma^2 > 0$ and Ψ is positive definite.

- (3 points) Calculate the covariance matrix $\mathbb{V}\left(\hat{\beta}\right)$ of the GLS estimator $\hat{\beta} = \left(X'\Psi^{-1}X\right)^{-1} X'\Psi^{-1}y$.

2. (6 points) We consider the models under assumptions (A1)-(A4)

$$y_i = x_i'\beta + z_i'\gamma + \varepsilon_i \qquad (B)$$

and

$$y_i = x_i'\beta + v_i \qquad (S)$$

(a) Assume model (S) is true, but we happen to estimate (B):

i. The estimate $\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix}$ for $\begin{pmatrix}\beta\\ \gamma\end{pmatrix}$ is unbiased. True or false (0,75 points)? Give an explanation in (at most) one sentence (0.75 points).

ii. The estimate $\begin{pmatrix}\hat{\beta}\\ \hat{\gamma}\end{pmatrix}$ for $\begin{pmatrix}\beta\\ \gamma\end{pmatrix}$ is efficient. True or false (0,75 points)? Give an explanation in (at most) one sentence (0.75 points).

(b) Assume model (B) is true, but we happen to estimate (S)

i. The estimate $\hat{\beta}$ for β is unbiased. True or false (0,75 points)? Give an explanation in (at most) one sentence (0.75 points).

ii. The estimate $\hat{\beta}$ for β is efficient. True or false (0,75 points)? Give an explanation in (at most) one sentence (0.75 points).

3. The p-value corresponding to testing $H_0: \ \beta = \beta_0$ is 0.075.

(a) (1 point) H_0 gets rejected at level 10%. True of false?

(b) (1 point) H_0 gets rejected at level 5%. True or false?

4. Assuming that a test with Type I error $\alpha = 0.10$ has been constructed to test $H_0: \ \beta = \beta_0$. Compare two different alternative hypotheses $\beta_0 < \beta^{(1)} < \beta^{(2)}$.

- (2 points) The inequality $\mathbb{P}\left(\text{reject } H_0 | \beta^{(1)} \text{ true}\right) > \mathbb{P}\left(\text{reject } H_0 | \beta^{(2)} \text{ true}\right)$ holds. True or false (1 point)? Give an explanation (1 point).