Advanced Econometrics I: Principles of Econometrics
Final Exam

16 December 2016

This exam has three pages. Good luck!

1 Gauß-Markov Theorem (12 points)

Prove the Gauß-Markov Theorem: The OLS estimator $b = (X'X)^{-1} X'y$ is the BLUE for β in the linear model $y = X\beta + \varepsilon$, and the variance of b is $\sigma^2 (X'X)^{-1}$.

1.1 Statement and definitions (4 points)

1. State the Gauß-Markov assumptions (you may assume that the matrix $X \in \mathbb{R}^{N \times K}$, $N \geq K$, is non-stochastic).
2. Explain each letter in the acronym "BLUE" and give definitions of the terms involved.

1.2 Part 1 of the proof (4 points)

1. Derive the conditions on the matrix D in the linear estimator $\breve{b} = Dy$ such that it satisfies property "U" in "BLUE".
   (a) What dimensions does D have?
2. Verify that the OLS estimator b satisfies the derived condition.

1.3 Part 2 of the proof (4 points)

1. Show that for matrices D, $X \in \mathbb{R}^{N \times K}$, I (the identity matrix) satisfying $DX = I$, the decompositition

$$DD' = \left((X'X)^{-1} X'\right) \left((X'X)^{-1} X'\right)' + \left(D - \left[(X'X)^{-1} X'\right]\right) \left(D - \left[(X'X)^{-1} X'\right]\right)$$

   holds.
2. Show that the OLS estimator satisfies "B" in the acronym BLUE.

2 F-test (12 points)

Consider the regression model

$$y = X_1\beta^{(1)} + X_2\beta^{(2)} + \varepsilon$$

satisfying assumptions (A1)-(A5). Furthermore, assume that $X_1 \in \mathbb{R}^{N\times(K-J)}$ and $X_2 \in \mathbb{R}^{N\times J}$ are non-stochastic. Derive the F-test for $H_0: \beta_{K-J+1} = \cdots = \beta_K = 0$, i.e. show that

$$\frac{\frac{\frac{1}{\sigma^2}\left(e'_{restr}e_{restr} - e'e\right)}{J}}{\frac{\frac{1}{\sigma^2}e'e}{N-K}} \sim F_{J,N-K}$$

where e is the OLS residual in the unrestricted model, and e_{restr} is the OLS residual under H_0, i.e. in the restricted model.

1. (3 points) Assuming $\frac{1}{\sigma^2}\left(e'_{restr}e_{restr} - e'e\right)$ and $\frac{1}{\sigma^2}e'e$ are independent, why is $f = \frac{\frac{\left(e'_{restr}e_{restr} - e'e\right)}{J}}{\frac{e'e}{N-K}}$ distributed as $F_{J,N-K}$?

2. (3 points) Project the equation $y = X_1 b^{(1)} + X_2 b^{(2)} + e$, where $b = \begin{pmatrix} b^{(1)} \\ b^{(2)} \end{pmatrix}$ is the OLS estimator for $\beta = \begin{pmatrix} \beta^{(1)} \\ \beta^{(2)} \end{pmatrix}$, on the orthogonal complement of the space spanned by the columns of X_1 and show that $e'_{restr}e_{restr} = \left(b^{(2)}\right)' \tilde{X}_2'\tilde{X}_2 b^{(2)} + e'e$. Explain your calculations.

3. (3 points) Show that under H_0 the equality $\frac{1}{\sigma^2}\left(e'_{restr}e_{restr} - e'e\right) = \frac{1}{\sigma^2}\left[\varepsilon' P_{\tilde{X}_2}\varepsilon\right]$ holds. (You may use the Frisch-Waugh Theorem, i.e. use the fact that $b^{(2)} = \left(\left(\tilde{X}_2\right)'\tilde{X}_2\right)^{-1}\left(\tilde{X}_2\right)' y$.) Explain your calculations.

4. (3 points) Show that $\frac{1}{\sigma^2}e'e$ is equal to $\frac{1}{\sigma^2}\varepsilon' M_X\varepsilon$. Then, show that $\frac{1}{\sigma^2}\varepsilon' M_X\varepsilon$ and $\frac{1}{\sigma^2}\left[\varepsilon' P_{\tilde{X}_2}\varepsilon\right]$ are independent.

   (a) Hint: In order to show that $\frac{1}{\sigma^2}\varepsilon' M_X\varepsilon$ and $\frac{1}{\sigma^2}\left[\varepsilon' P_{\tilde{X}_2}\varepsilon\right]$ are independent, start by showing that $M_X\varepsilon$ and $P_X\varepsilon$ are uncorrelated, then use properties of normal distributions to show that the quadratic forms above are independent.

3 Short Questions (12 points + 2 points BONUS)

We consider the linear regression model

$$y = X\beta + \varepsilon$$

where $X \in \mathbb{R}^{N\times K}$ is non-stochastic, $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{V}(\varepsilon) = \sigma^2\Psi$, where $\sigma^2 > 0$ and Ψ is positive definite. Furthermore, $\frac{1}{N}(X'X) \xrightarrow{N\to\infty} \Sigma_{xx}$ is positive-definite.

1. (3 points) Calculate the covariance matrix $\mathbb{V}(b)$ of the OLS estimator $b = (X'X)^{-1}X'y$.
2. (1 point) The OLS estimator is unbiased in this model. True or false?
3. (1 point) The OLS estimator is consistent in this model. True or false?

Consider, under the assumptions above, the case of heteroskedasticity, i.e. $\mathbb{V}(\varepsilon) = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_N^2 \end{pmatrix}$.

1. (1 point) The weighted least squares (WLS) estimator gives a different weight to each column of X. True or false?
2. (1 point) The matrix $\left(\frac{1}{N}\sum_{i=1}^N e_i^2 x_i x_i'\right)$ in $\hat{\mathbb{V}}(b) = \frac{1}{N}\left(\frac{1}{N}X'X\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N e_i^2 x_i x_i'\right)\left(\frac{1}{N}X'X\right)^{-1}$ is a consistent estimator for $\left(\frac{1}{N}\sum_{i=1}^N \sigma_i^2 x_i x_i'\right)$. True or false?
3. (1 point) How many free parameters are in $\left(\frac{1}{N}\sum_{i=1}^N \sigma_i^2 x_i x_i'\right)$? Why is this advantageous compared to estimating σ_i^2 for all i (give a numerical example with $K = 5$ and $N = 200$)?

Consider the $H_0:\ R\beta = q$. Let $\hat{\theta}$ be the unrestricted MLE, and let $\tilde{\theta}$ be the restricted MLE.

1. (1 point) The unrestricted MLE $\hat{\theta}$ is unbiased, asymptotically efficient, and asymptotically normal. True or false?
2. (1 point) Define the Wald test statistic and give a geometric interpretation. Which model needs to be estimated?
3. (1 point) Define the LR test statistic and give a geometric interpretation. Which model needs to be estimated?
4. (2 points) Define the LM test statistic and give a geometric interpretation. Which model needs to be estimated?
5. (1 point) The LR test is sensitive with respect to the formulation of the restriction. True or false?